Ismétlés:

$\mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(e^{kt}) = \frac{1}{s-k} \qquad s > k$

$\mathcal{L}(\cosh(kt)) = \frac{s}{s^2 - k^2} \qquad \mathcal{L}(\sinh(kt)) = \frac{k}{s^2 - k^2}$

$\mathcal{L}(\cos(kt)) = \frac{s}{s^2 + k^2} \qquad \mathcal{L}(\sin(kt)) = \frac{k}{s^2 + k^2}$

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

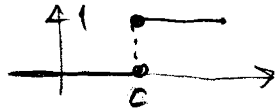
$\mathcal{L}[e^{at} f(t)] = F(s-a)$, ahol $F(s) = \mathcal{L}(f(t))$ $\Longrightarrow$ csillapítási tétel

$\mathcal{L}(e^{at} \cos kt) = \frac{s-a}{(s-a)^2 + k^2} \qquad \mathcal{L}(e^{at} \sin kt) = \frac{k}{(s-a) + k^2}$

$\mathcal{L}(f') = -f(0) + s\mathcal{L}(f)$

$\mathcal{L}(f'') = s^2 \mathcal{L}(f) - sf(0) - f'(0)$

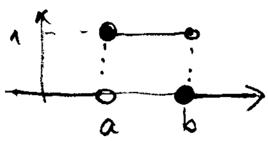
Heaviside:



$H_c(t) \qquad \mathcal{L}(H_c) = \frac{e^{-sc}}{s} \qquad s > 0$

Eltolási tétel: $\mathcal{L}[H_c(t) f(t-c)] = e^{-sc} \cdot \mathcal{L}(f)$

Négyszögjel:



$f(t) = H_a(t) - H_b(t)$

$\mathcal{L}(f) = \mathcal{L}(H_a) - \mathcal{L}(H_b) = \frac{e^{-sa} - e^{-sb}}{s}$.

Átviteli függvény:

Hatás: $u(t)$ Válasz: $v(t)$

$a_n v^{(n)} + \dots + a_1 v' + a_0 v = b_m u^{(m)} + \dots + b_1 u' + b_0 u \Longrightarrow$

$a_n s^n V(s) + \dots + a_1 s V(s) + a_0 V(s) = b_m s^m U(s) + \dots + b_1 s U(s) + b_0 U(s)$

$$Y(s) = \frac{V(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$ átviteli fgv.

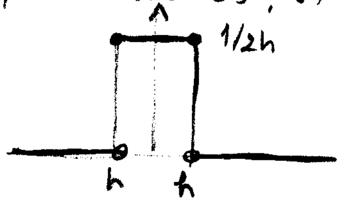
Vizsgálójelek:

Mi a Heaviside deriváltja?

$$\mathcal{L}(H_c'(t)) = s\cdot \underbrace{\mathcal{L}(H_c)}_{} - H_c(0) = s\cdot \overbrace{\frac{e^{-sc}}{s}}^{\mathcal{L}(H_c)} = e^{-sc} \qquad \text{Mi ez?}$$

Dirac delta (pillanatnyi impulzus, áramlökés, stb.)

$$\delta_h(t) = \begin{cases} 1/2h & -h \le t \le h \\ 0 & \text{egyébként} \end{cases}$$



Teljesül: $\int_{-\infty}^{\infty} \delta_h(t)\,dt = 1$

$h \to 0$ határesetben is teljesül.

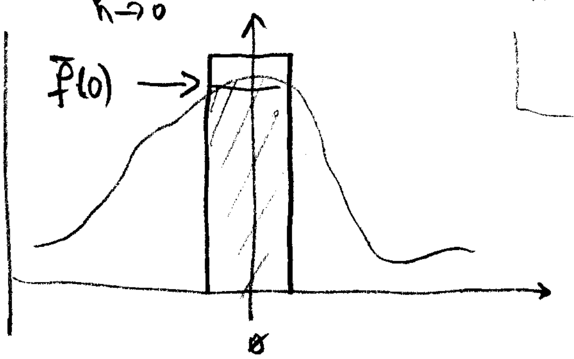
Határesetben: $\delta(t) = \lim_{h\to 0} \delta_h(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$ ahol $\int \delta(t)\,dt = 1$

Ha folytonos lenne, teljesülne, hogy

$$\lim_{h\to 0} \int \delta_h(t)\,dt = \int \lim_{h\to 0} \delta_h(t)\,dt = \int \delta(t)\,dt = 1$$

Megkívánjuk:

$$\lim_{h\to 0} \int \delta_h(t) f(t)\,dt = \int \underbrace{\lim_{h\to 0} \delta_h(t)}_{\delta(t)} f(t)\,dt = f(0).$$



Bal oldal:

$$\int_{-\infty}^{\infty} \delta_h(t) f(t)\,dt = \frac{1}{2h}\int_{-h}^{h} f(t)\,dt = \bar{f}(0) =$$

$$= f(\xi) \qquad -h < \xi < h$$

Ha $h \to 0$, akkor $\bar{f}(0) \to f(0)$.

$\delta(t)$ kiveszi a fgv. 0-beli értékét az integrálban.

Eltolás:

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta(t-c)\,dt = f(c)}$$

Dirac δ Laplace-a

$$\mathcal{L}(\delta(t-c)) = \int_0^\infty e^{-st}\delta(t-c) = \int_{-\infty}^{\infty} e^{-st}\delta(t-c) = \underline{\underline{e^{-sc}}}$$

Tehát a Heaviside deriváltja a Dirac delta!

Speciális eset:

$c = 0$

$$\boxed{\begin{aligned}\mathcal{L}(\delta(t)) &= 1\\ \mathcal{L}(H_0(t)) &= \frac{1}{s}\end{aligned}}$$

$H_0(t) = 1(t)$

Hatás: $u = \delta(t)$ **Válasz**: súlyfüggvény: $v = w(t)$

$$Y(s) = \frac{V(s)}{U(s)} = \frac{W(s)}{1}$$ átviteli fgv. a súlyfgv. Laplace tr.

Hatás: $u = 1(t)$ **Válasz**: átmeneti fgv: $v = v_a(t)$

$$\underline{\underline{V_a(s)}} = U(s)\cdot Y(s) = \frac{1}{s}\cdot Y(s) = \underline{\underline{\frac{1}{s}W(s)}}$$

s-sel való osztás integrálásnak felel meg!

$F(t) = \int_0^t f(\tau)\,d\tau$ primitív fgv.

$$\underline{\underline{\mathcal{L}(F(t))}} = \int_0^\infty \underset{u}{F(t)}\,\underset{v'}{e^{-st}}\,dt = \left[F(t)\left(-\frac{1}{s}\right)e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt =$$

$u' = f \qquad v = -\frac{1}{s}e^{-st} \qquad F(0) = 0$

$$= \underline{\underline{\frac{1}{s}\mathcal{L}(f)}}$$

Tehát: $v_a(t) = \int_0^t w(t)\,d\tau$ átm. fgv a súlyfgv. integrálja

ill. $w(t) = dv_a(t)/dt$

Példa

$$T \cdot \dot{v} + v = K \cdot u(t) \qquad v(0) = 0$$

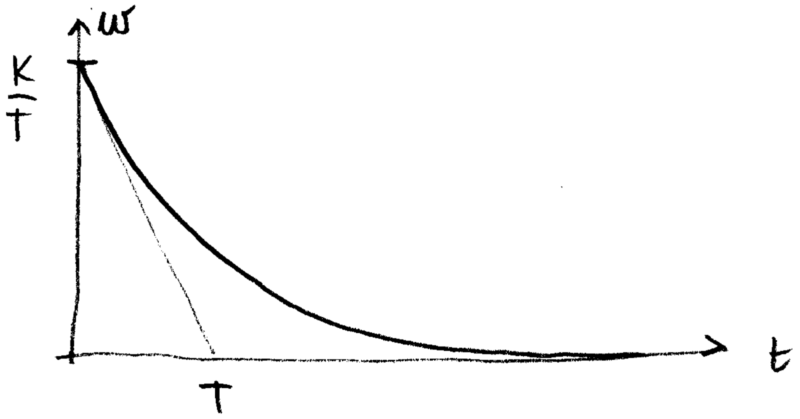
$$T\left(sV(s) - v(0)\right) + V(s) = K \cdot U(s)$$

$$Y(s) = \frac{V(s)}{U(s)} = \frac{K}{1+Ts} \Rightarrow V(s) = K \cdot \frac{1}{1+Ts} V(s)$$

(a) $u(t) = \delta(t) \Rightarrow U(s) = 1$

$$V(s) = K \frac{1}{1+Ts} = \frac{K}{T} \frac{1}{\frac{1}{T}+s} = W(s)$$

$$\mathcal{L}^{-1}(V) = K \mathcal{L}^{-1}\left(\frac{1}{1+Ts}\right) = \frac{K}{T} \mathcal{L}^{-1}\left(\frac{1}{\frac{1}{T}+s}\right) = \frac{K}{T} e^{-t/T} = v(t) = w(t)$$



(b) $u(t) = 1(t) \Rightarrow U(s) = \frac{1}{s}$

$$V(s) = K \frac{1}{s(1+Ts)} = K\left(\frac{A}{1+Ts} + \frac{B}{s}\right)$$

$$\left.\begin{aligned} s=1 \quad & \frac{1}{1+T} = \frac{A}{1+T} + B \\ s=-1 \quad & \frac{1}{-(1-T)} = \frac{A}{1-T} - B \end{aligned}\right\} + \qquad B = \frac{1}{1+T} + \frac{T}{1+T} = \frac{1+T}{1+T} = 1$$

$$\frac{1}{1+T} + \frac{1}{T-1} = A\left(\frac{1}{T+1} - \frac{1}{T-1}\right) \qquad A = -T$$

$$\frac{T-\cancel{1}+T+\cancel{1}}{\cancel{1-T^2}} = A\,\frac{\cancel{T}-1-\cancel{T}+1}{\cancel{1-T^2}} = -2A$$

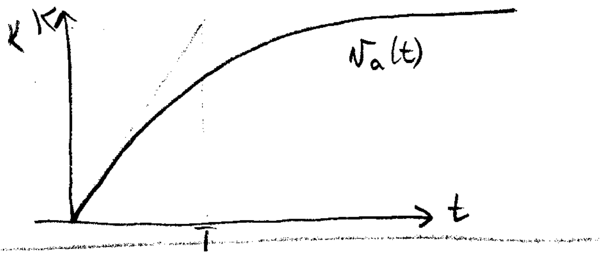
$2T$

folyt

$$V(s) = K\left(-\frac{T}{1+Ts} + \frac{1}{s}\right) =$$

$$\mathcal{L}^{-1}(V) = -KT\cdot\underbrace{\mathcal{L}^{-1}\left(\frac{1}{1+Ts}\right)}_{\frac{1}{T}\cdot e^{-t/T}} + K\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s}\right)}_{1} = K\left(-e^{-t/T}+1\right) =$$

$$\boxed{v_a(t) = K\left(1 - e^{-t/T}\right)}$$



Ellenőrzés: $v_a'(t) = K\frac{1}{T}e^{-t/T} = w(t)$

(c) $u(t) = t\cdot 1(t) \Rightarrow U(s) = \frac{1}{s^2}$

$$V(s) = K\frac{1}{s^2(1+Ts)} \qquad \mathcal{L}^{-1}(V) = K\left(Te^{-t/T} + t - T\right)$$

$$\frac{1}{s^2(1+Ts)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{1+Ts}$$

$s=1$ $\quad \frac{1}{1+T} = A + B + \frac{C}{1+T}$

$s=-1$ $\quad \frac{1}{1-T} = -A + B + \frac{C}{1-T}$

$s=2$ $\quad \frac{1}{4(1+2T)} = \frac{A}{2} + \frac{B}{4} + \frac{C}{1+2T}$ $\quad /2$

$+$ $\quad \frac{1}{1+T} + \frac{1}{1-T} = 2B + C\left(\frac{1}{1+T} + \frac{1}{1-T}\right)$

$$\frac{1\cancel{-T} + 1\cancel{+T}}{1-T^2}$$

$$\frac{2}{\cancel{1-T^2}} = \frac{2B(1-T^2)}{\cancel{1-T^2}} + C\frac{2}{\cancel{1-T^2}}$$

$$1 = B(1-T^2) + 2C$$

$$1 = 2A(1+2T) + B(1+2T) + 4C \quad \}+$$

$$\frac{2(1+2T)}{1-T} = -2A(1+2T) + 2B(1+2T) + \frac{2C(1+2T)}{1-T}$$

ELHISSZÜK

d) $u(t) = \sin\alpha t \implies U(s) = \frac{\alpha}{s^2+\alpha^2}$

$$V(s) = \frac{K}{T}\,\frac{1}{s+\frac{1}{T}}\,\frac{\alpha}{s^2+\alpha^2}$$

Kell olyan fgv. szorzata, amik benne vannak a táblázatba $\Rightarrow$

$\Rightarrow$

Konvolúció

Def.: $(f*g)(t) := \int_0^t f(t-u)\,g(u)\,du \qquad t>0$

Konvolúciós tétel:

$$\boxed{\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)}$$

$F(s) = \frac{1}{a+s} \qquad G(s) = \frac{\alpha^2}{s^2+\alpha^2}$

$f(t) = e^{-\alpha t} \qquad g(t) \quad \sin\alpha t$

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) = \mathcal{L}^{-1}\left(\mathcal{L}(F*G)\right) = F*G = \int_0^t f(t-u)\,g(u)\,du =$$

$$= \int_0^t \underset{\dot{x}}{e^{-a(t-u)}}\,\underset{y}{\sin\alpha u}\,du = \left[+\frac{1}{a}e^{-a(t-u)}\sin\alpha u\right]_0^t - \frac{\alpha}{a}\int_0^t e^{-a(t-u)}\cos\alpha u\,du =$$

$x = +\frac{1}{a}e^{-a(t-u)} \qquad y' = \alpha\cdot\cos(\alpha u)$

$$= \frac{1}{a}\sin\alpha t - \frac{\alpha}{a}\int_0^t \underset{\dot{x}}{e^{-a(t-u)}}\,\underset{y}{\cos(\alpha u)}\,du = \frac{1}{a}\sin\alpha t - \frac{\alpha}{a}\left\{\left[\frac{1}{a}e^{-a(t-u)}\cos\alpha u\right]_0^t + \right.$$

$y' = -\alpha\cdot\sin\alpha u \qquad \frac{1}{a}\cdot\cos\alpha t - \frac{1}{a}e^{-at}$

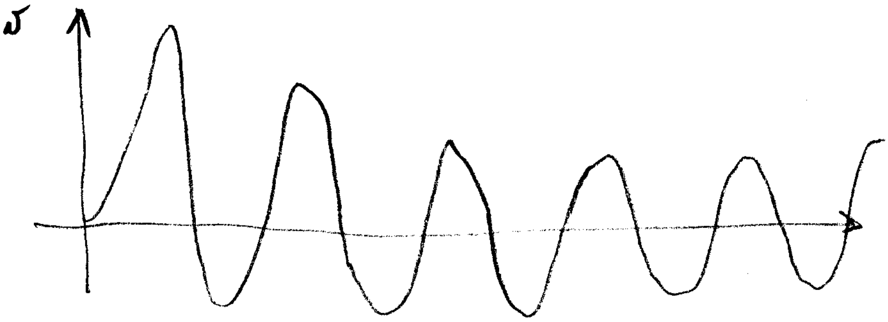
$$\left.\underbrace{+\frac{\alpha}{a}\int_0^t e^{-a(t-u)}\sin\alpha u\,du}_{I}\right\}$$

$$I = \frac{1}{a}\sin\alpha t - \frac{\alpha}{a^2}\cos\alpha t + \frac{\alpha}{a^2}e^{-at} - \frac{\alpha^2}{a^2} I$$

$$I = \frac{a}{\alpha^2+a^2}\sin\alpha t - \frac{\alpha}{\alpha^2+a^2}\cos\alpha t + \frac{\alpha}{\alpha^2+a^2}e^{-at}$$

$a = \frac{1}{T}$ és normi $\frac{K}{T}$ -vel $\Rightarrow$

$$v(t) = \frac{K}{1+\alpha^2T^2}\left(\sin\alpha t - \alpha T\cos\alpha t + \alpha T e^{-t/T}\right)$$



(e) $u(t) = 1(t-\tau) = H_\tau(t) \Rightarrow U(s) = \frac{1}{s}e^{-\tau s}$

$$V(s) = K\frac{1}{s(1+Ts)}e^{-\tau s}$$

(b) feladat

Eltolási szabály:

$$e^{-\tau s}\cdot\mathcal{L}(v_a(t)) = \mathcal{L}\Big(\underbrace{H_\tau(t)\cdot v_a(t-\tau)}\Big)$$