1 Variables and Functions

1.1 INTRODUCTION

The usual whole numbers, integers such as $1, 2, 3, 4 \ldots$, are usually referred to as Arabic numerals. It seems, however, that the basic decimal counting system was first developed in India, as it was demonstrated in an Indian astronomic calendar which dates from the third century AD. This system, which was composed of nine figures and the zero, was employed by the Arabs in the ninth century. The notation is basically that of the Arabic language and it was the Arabs who introduced the system in Europe at the beginning of the eleventh century.

In Europe the notion of the zero evolved slowly in various forms. Eventually, probably to express debts, it was found necessary to invent negative integers. The requirements of trade and commerce lead to the use of fractions, as ratios of whole numbers. However, it is obviously more convenient to express fractions in the form of decimals. The ensemble of whole numbers and fractions (as ratios of whole numbers) is referred to as rational numbers. The mathematical relation between decimal and rational fractions is of importance, particularly in modern computer applications.

As an example, consider the decimal fraction $x = 0.616161\cdots$. Multiplication by 100 yields the expression $100x = 61.6161\cdots = 61 + x$ and thus, $x = 61/99$, is a rational fraction. In general, if a decimal expression contains an infinitely repeating set of digits (61 in this example), it is a rational number. However, most decimal fractions do not contain a repeating set of digits and so are not rational numbers. Examples such as $\sqrt{3} = 1.732051\cdots$ and $\pi = 3.1415926536\cdots$ are irrational numbers.* Furthermore, the logarithms and trigonometric functions of most arguments are irrational numbers.

*A mnemonic for π based on the number of letters in words of the English language is quoted here from "the Green Book", Ian Mills, *et al.* (eds), "Quantities, Units and Symbols in Physical Chemistry", Blackwell Scientific Publications, London (1993):

'How I like a drink, alcoholic of course, after the heavy lectures involving quantum mechanics!'

In practice, in numerical calculations with a computer, both rational and irrational numbers are represented by a finite number of digits. In both cases, then, approximations are made and the errors introduced in the result depend on the number of significant figures carried by the computer – the machine precision.* In the case of irrational numbers such errors cannot be avoided.

The ensemble of rational and irrational numbers are called real numbers. Clearly, the sum, difference and product of two real numbers is real. The division of two real numbers is defined in all cases but one – division by zero. Your computer will spit out an error message if you try to divide by zero!

1.2 FUNCTIONS

If two real variables are related such that, if a value of x is given, a value of y is determined, y is said to be a function of x. Thus, values may be assigned to x, the independent variable, leading to corresponding values of y, the dependent variable.

As an example, consider one mole of a gas at constant temperature. The volume V is a function of the applied pressure P. This relation can be expressed mathematically in the form

$$V = f(P), \tag{1}$$

or, $V = V(P)$. Note that to complete the functional relationship, the nature of the gas, as well as the temperature T, must be specified. A physical chemist should also insist that the system be in thermodynamic equilibrium.

In the case of an ideal gas, the functional relationship of Eq. (1) becomes

$$V = \frac{C(T)}{P}, \tag{2}$$

where $C(T)$ is a positive constant which is proportional to the absolute temperature T. Clearly, the roles of V and P can be reversed leading to the relation

$$P = \frac{C(T)}{V}. \tag{3}$$

The question as to which is the independent variable and which is the dependent one is determined by the way in which the measurements are made and, mathematically, on the presentation of the experimental data.

*Note that the zero is a special case, as its precision is not defined. Normally, the computer automatically uses the precision specified for other numbers.

Suppose that a series of measurements of the volume of a gas is made, as the applied pressure is varied. As an example, the original results obtained by Boyle* are presented as in Table 1.

In this case V is a function of P, but it is not continuous. It is the discrete function represented by the points shown in Fig. 1a. It is only the mathematical function of Eq. (2) that is continuous. If, from the experimental data, it is of interest to calculate values of V at intermediate points, it is necessary to estimate them with the use of, say, linear interpolation, or better, a curve-fitting procedure. In the latter case the continuous function represented by Eq. (2)

Table 1 Volume of a gas as a function of pressure.

V, Volume[a]	P, Pressure (inches Hg)	$1/V$
12	29.125	0.0833
10	35.3125	0.1
8	44.1875	0.125
6	58.8125	0.1667
5	70.6875	0.2
4	87.875	0.25
3	116.5625	0.3333

[a]Measured distance (inches) in a tube of constant diameter.

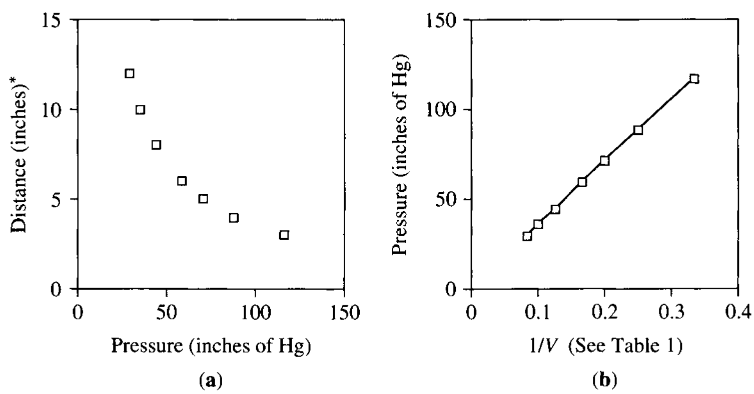

Fig. 1 Volume of a gas (expressed as distance in a tube of constant diameter) *versus* pressure (**a**). Pressure as a function of reciprocal volume (**b**).

*Robert Boyle, Irish physical chemist (1627–1691) would normally be employed. These questions, which concern the numerical treatment of data, will be considered in Chapter 13.

In Boyle's work the pressure was subsequently plotted as a function of the reciprocal of the volume, as calculated here in the third column of Table 1. The graph of P *vs.* $1/V$ is shown in Fig. 1b. This result provided convincing evidence of the relation given by Eq. (3), the mathematical statement of Boyle's law. Clearly, the slope of the straight line given in Fig. 1b yields a value of $C(T)$ at the temperature of the measurements [Eq. (3)] and hence a value of the gas constant R. However, the significance of the temperature was not understood at the time of Boyle's observations.

In many cases a series of experimental results are not associated with a known mathematical function. In the following example Miss X weighed herself each morning beginning on the first of February. These data are presented graphically as shown in Fig. 2. Here interpolated points are of no significance, nor is extrapolation. By extrapolation Miss X would weigh nearly nothing in a year-and-a-half or so. However, as the data do exhibit a trend over a relatively short time, it is useful to employ a curve-fitting procedure. In this example Miss X might be happy to conclude that on the average she lost 0.83 kg per week during this period, as indicated by the slope of the straight line in Fig. 2.

Now reconsider the function given by Eq. (3). It has the form of a hyperbola, as shown in Fig. 3. Different values of $C(T)$ lead to other members of the family of curves shown. It should be noted that this function is antisymmetric with respect to the inversion operation $V \to -V$ (see Chapter 8). Thus, P is said to be an odd function of V, as $P(V) = -P(-V)$.

It should be evident that the negative branches of P *vs.* V shown in Fig. 3 can be excluded. These branches of the function are correct mathematically,

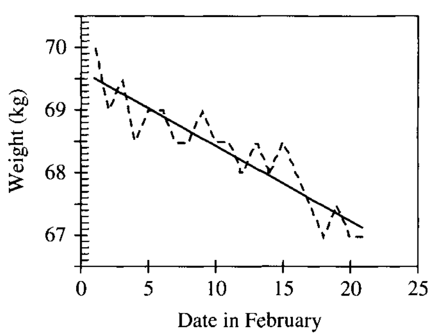

Fig. 2 Miss X's weight as a function of the date in February. The straight line is obtained by a least-squares fit to the experimental data (see Chapter 13).

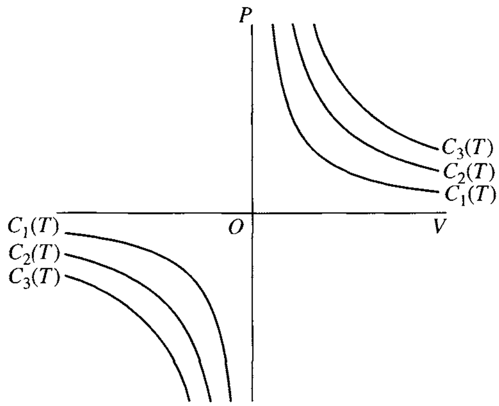

Fig. 3 Pressure *versus* volume [Eq. (3)], with $C_3(T) > C_2(T) > C_1(T)$.

but are of no physical significance for this problem. This example illustrates the fact that functions may often be limited to a certain domain of acceptability. Finally, it should be noted that the function $P(V)$ presented in Fig. 3 is not continuous at the origin ($V = 0$). Therefore, from a physical point of view the function is only significant in the region $0 < V < \infty$. Furthermore, physical chemists know that Eqs. (2) and (3) do not apply at high pressures because the gas is no longer ideal.

As $C(T)$ is a positive quantity, Eq. (3) can be written in the form

$$ln\ P = ln\ C - ln\ V. \tag{4}$$

Clearly a plot of $ln\ P$ *vs.* $ln\ V$ at a given (constant) temperature yields a straight line with an intercept equal to $ln\ C$. This analysis provides a convenient graphical method of determining the constant C.

It is often useful to shift the origin of a given graph. Thus, for the example given above consider that the axes of V and P are displaced by the amounts v and p, respectively. Then, Eq. (3) becomes

$$P - p = \frac{C(T)}{V - v}, \tag{5}$$

and the result is as plotted in Fig. 4. The general hyperbolic form of the curves has not been changed, although the resulting function $P(V)$ is no longer odd – nor is it even.

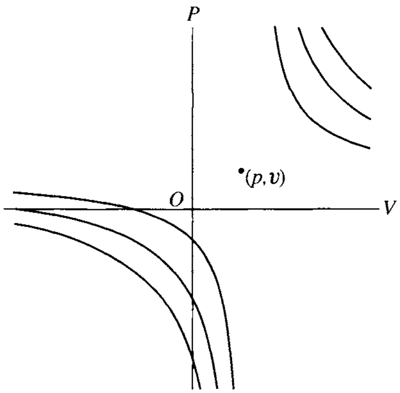

Fig. 4 Plots of Eq. (5) for given values of v and p.

1.3 CLASSIFICATION AND PROPERTIES OF FUNCTIONS

Functions can be classified as either algebraic or transcendental. Algebraic functions are rational integral functions or polynomials, rational fractions or quotients of polynomials, and irrational functions. Some of the simplest in the last category are those formed from rational functions by the extraction of roots. The more elementary transcendental functions are exponentials, logarithms, trigonometric and inverse trigonometric functions. Examples of these functions will be discussed in the following sections.

When the relation $y = f(x)$ is such that there is only one value of y for each acceptable value of x, $f(x)$ is said to be a single-valued function of x. Thus, if the function is defined for, say, $x = x_1$, the vertical line $x = x_1$ intercepts the curve at one and only one point, as shown in Fig. 5. However, in many cases a given value of x determines two or more distinct values of y.

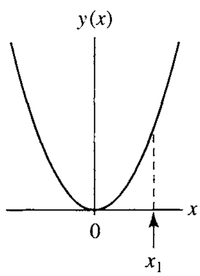

Fig. 5 Plot of $y = x^2$.

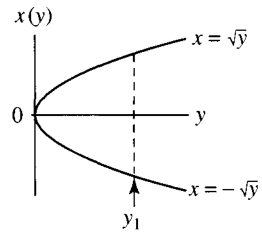

Fig. 6 Plot of $x = \pm\sqrt{y}$.

The curve shown in Fig. 5 can be represented by

$$y = x^2, \tag{6}$$

where y has the form of the potential function for a harmonic oscillator (see Chapter 5). This function is an even function of x, as $y(x) = y(-x)$. Clearly, y is a single-valued function of x. Now, if Eq. (6) is rewritten in the equivalent form

$$x^2 = y, \quad y \geq 0 \tag{7}$$

it defines a double-valued function whose branches are given by $x = \sqrt{y}$ and $x = -\sqrt{y}$. These branches are the upper and lower halves of the parabola shown in Fig. 6. It should be evident from this example that to obtain a given value of x, it is essential to specify the particular branch of the (in general) multiple-valued function involved. This problem is particularly important in numerical applications, as carried out on a computer (Don't let the computer choose the wrong branch!).

1.4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

If $y = f(x)$ is given by

$$y = a^x \tag{8}$$

$y = f(x)$ is an exponential function. The independent variable x is said to be the argument of f. The inverse relation, the logarithm, can then be defined by

$$x = log_a\, y \tag{9}$$

and a is called the base of the logarithm. It is clear, then, that $log_a\, a = 1$ and $log_a\, 1 = 0$. The logarithm is a function that can take on different values depending on the base chosen. If $a = 10$, log_{10} is usually written simply as log. A special case, which is certainly the most important in physics and chemistry, as well as in pure mathematics, is that with $a = e$. The quantity e, which serves as the base of the natural or Naperian* logarithm, $log_e \equiv ln$, can be defined by the series†

$$y = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{10}$$

It should be noted that the derivative of Eq. (10), taken term by term, is given by

$$\frac{dy}{dx} = 0 + 1 + x + \frac{1}{2!}x^2 + \cdots = e^x. \tag{11}$$

Thus, $(d/dx)e^x = e^x$ and here the operator d/dx plays the role of the identity with respect to the function $y = e^x$.‡ It will be employed in the solution of differential equations in Chapter 5.

Consider, now, the function $f(n) = (1 + 1/n)^n$. It is evaluated in Table 2 as a function of n, where it is seen that it approaches the value of $e \equiv lim_{n\to\infty}(1 + 1/n)^n = 2.7182818285\cdots$, an irrational number, as n becomes infinite. For simplicity, it has been assumed here that n is an integer, although it can be shown that the same limiting value is obtained for noninteger values of n. The identification of e with that employed in Eq. (10) can be made by

Table 2 Evaluation of $e \equiv lim_{n\to\infty}(1 + 1/n)^n$.

n	$f(n) = (1 + 1/n)^n$
1	2.000
2	2.250
5	2.489
10	2.594
20	2.653
50	2.691
100	2.705
1000	2.717
10 000	2.718
∞	2.7182818285

*John Napier or Neper, Scottish mathematician (1550–1617).

†The factorial $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$ (with $0! = 1$) has been introduced in Eq. (10). See also Section 4.5.4.

‡Note that e^x is often written $exp\, x$.

application of the binomial theorem (see Section 2.10). The functions e^x and $ln_e\, x$ are illustrated in Figs. 7 and 8, respectively.

As indicated above, the two logarithmic functions ln and log differ in the base used. Thus, if $y = e^x = 10^z$,

$$z = log\ y \tag{12}$$

and

$$x = ln\ y. \tag{13}$$

Then,

$$ln\ y = ln(e^x) = ln(10^z) \tag{14}$$

and, as $ln\ e = 1$ and $ln\ 10 = 2.303$,

$$ln\ y = 2.303\, log\ y. \tag{15}$$

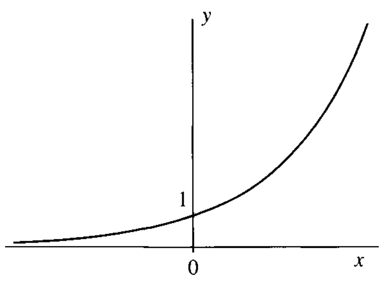

Fig. 7 Plot of $y(x) = e^x$.

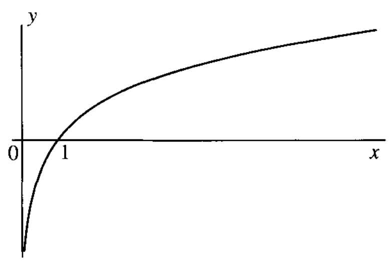

Fig. 8 Plot of $y(x) = ln_e\, x$.

The numerical factor 2.303 (or its reciprocal) appears in many formulas of physical chemistry and has often been the origin of errors in published scientific work. It is evident that these two logarithmic functions, *ln* and *log*, must be carefully distinguished.

It was shown above that the derivative of e^x is equal to e^x. Thus, if $x = ln\, y$, $y = e^x$ and

$$\frac{dy}{dx} = e^x = y. \tag{16}$$

Then,

$$\frac{dx}{dy} = \frac{d}{dy}(ln\, y) = \frac{1}{y} \tag{17}$$

and, as $dx = dy/y$,

$$x = \int \frac{1}{y} dy = ln\, y + C, \tag{18}$$

where C is here the constant of integration (see Chapter 3).

1.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

As an example of the use of the exponential and logarithmic functions in physical chemistry, consider a first-order chemical reaction, such as a radioactive decay. It follows the rate law

$$-\frac{d[A]}{dt} = k[A], \tag{19}$$

where [A] represents the concentration of reactant A at time t. With the use of Eq. (18) this expression can be integrated to yield

$$-ln[A] = kt + C, \tag{20}$$

where C is a constant.* The integration constant C can only be evaluated if additional data are available. Usually the experimentalist measures at a given time, say t_0, the concentration of reactant, $[A]_0$. This relation, which constitutes an initial condition on the differential equation, Eq. (19), allows the integration constant C to be evaluated. Thus, $[A] = [A]_0$ at $t = t_0$, and

$$ln\, \frac{[A]}{[A]_0} = -kt. \tag{21}$$

*The indefinite integral is discussed in Section 3.1.

This expression can of course be written in the exponential form, *viz.*,

$$[A] = [A]_0 e^{-kt}, \tag{22}$$

the result that is plotted in Fig. 9.

In the case of radio-active decay the rate is often expressed by the half-life, namely, the time required for half of the reactant to disappear. From Eq. (22) the half-life is given by $t_{1/2} = (ln\ 2)/k$.

As a second example, consider the absorption of light by a thin slice of a given sample, as shown in Fig. 10. The intensity of the light incident on the sample is represented by I_0, while I is the intensity at a distance x. Following Lambert's law,* the decrease in intensity is given by

$$-\mathrm{d}I = \alpha I \mathrm{d}x, \tag{23}$$

where α is a constant. Integration of Eq. (23) leads to

$$-\ln I = \alpha x + C. \tag{24}$$

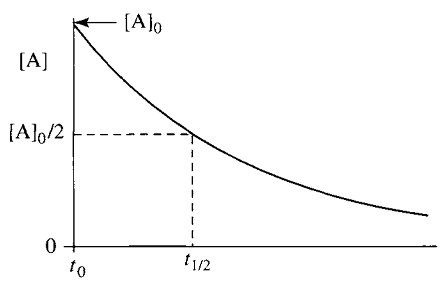

Fig. 9 Exponential decay of reactant in a first-order reaction.

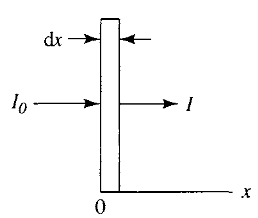

Fig. 10 Transmission of light through a thin slice of sample.

*Jean-Henri Lambert, French mathematician (1728–1777).

Here again, certain conditions must be imposed on the general solution of Eq. (24) to evaluate the constant of integration. They are in this case referred to as the boundary conditions. Thus if $I = I_0$ at $x = 0$, $C = -\,ln\, I_0$ and Eq. (24) becomes

$$ln\,\frac{I}{I_0} = -\alpha x. \tag{25}$$

For a sample of thickness $x = \ell$, the fraction of light transmitted is given by

$$\frac{I}{I_0} = e^{-\alpha\ell}. \tag{26}$$

The integration of Eq. (23) can also be carried out between limits (see Chapter 3), in the form

$$-\int_{I_0}^{I} \frac{\mathrm{d}I}{I} = \alpha \int_0^{\ell} \mathrm{d}x. \tag{27}$$

The result is then

$$ln\,\frac{I}{I_0} = -\alpha\ell \tag{28}$$

or

$$log\,\frac{I}{I_0} = -\left(\frac{\alpha}{2.303}\right)\ell. \tag{29}$$

Thus, the extinction coefficient, as usually defined in analytical spectroscopy, includes the factor 2.303 in the denominator. It should be apparent that the light intensity decreases exponentially within the sample, by analogy with the decrease in reactant in the previous example [Fig. (9)].

1.6 COMPLEX NUMBERS

Consider the relation $z = x + iy$, where x and y are real numbers and i has the property that $i^2 = -1$. The variable z is called a complex number, with real part x and imaginary part y. Thus, $\Re e[z] = x$ and $\Im m[z] = y$. It will be shown in Chapter 8 that the quantities $i^0 = 1, i^1 = i, i^2 = -1$ and $i^3 = -i$ form a group, a cyclic group of order four.

Two complex numbers which differ only in the sign of their imaginary parts are called complex conjugates – or simply conjugates. Thus, if $z = x + iy$, $z^\star = x - iy$ is its complex conjugate, which is obtained by replacing i by $-i$. Students are usually introduced to complex numbers as solutions to certain quadratic equations, where the roots always appear as conjugate pairs. It should be noted that in terms of absolute values $|z| = |z^{\star}| = \sqrt{x^2 + y^2}$, which is sometimes called the modulus of z.

It is often convenient to represent complex numbers graphically in what is referred to as the complex plane.* The real numbers lie along the x axis and the pure imaginaries along the y axis. Thus, a complex number such as $3 + 4i$ is represented by the point (3,4) and the locus of points for a constant value of $r = |z|$ is a circle of radius $|z|$ centered at the origin, as shown in Fig. 11. Clearly, $x = r\,cos\,\varphi$ and $y = r\,sin\,\varphi$, and in polar coordinates

$$z = x + iy = re^{i\varphi} \tag{30}$$

$$= r(cos\,\varphi + i\,sin\,\varphi). \tag{31}$$

Then,

$$e^{i\varphi} = cos\,\varphi + i\,sin\,\varphi, \tag{32}$$

which is the very important relation known as Euler's equation.† It should be emphasized here that the exponential functions of both imaginary and real arguments are of extreme importance. They will be discussed in some detail in Chapter 11 in connection with the Fourier and Laplace transforms, respectively.

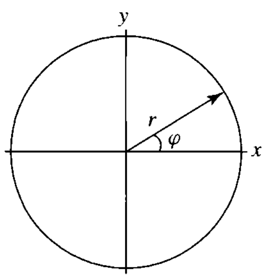

Fig. 11 Circle of radius $r = |z|$ in the complex plane.

*This system of representing complex numbers was developed by Jean-Robert Argand, Swiss mathematician (1768–1822), among others, near the beginning of the 19th century.

†Leonhard Euler, Swiss mathematician (1707–1783). This relation is sometimes attributed to Abraham De Moivre, British mathematician (1667–1754).

1.7 CIRCULAR TRIGONOMETRIC FUNCTIONS

The exponential function was defined in Eq. (10) terms of an infinite series. By analogy, the left-hand side of Eq. (32) can be expressed in the form

$$e^{i\varphi} = 1 + \frac{i\varphi}{1!} + \frac{(i\varphi)^2}{2!} + \cdots + \frac{(i\varphi)^n}{n!} + \cdots, \tag{33}$$

which can be separated into its real and imaginary parts, *viz.*

$$\Re e[e^{i\varphi}] = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{5!} - \cdots = cos\,\varphi \tag{34}$$

and

$$\Im m[e^{i\varphi}] = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots = sin\,\varphi. \tag{35}$$

Comparison with Eq. (32) yields the last equalities in Eqs. (34) and (35). The infinite series in these two equations are often taken as the fundamental definitions of the cosine and sine functions, respectively. The equivalent expressions for these functions,

$$cos\,\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \tag{36}$$

and

$$sin\,\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \tag{37}$$

can be easily derived from Eqs. (33–35). Alternatively, they can be used as definitions of these functions. The functions $cos\,\varphi$ and $sin\,\varphi$ are plotted *versus* φ expressed in radians in Figs. 12a and 12b, respectively. The two curves have the same general form, with a period of 2π, although they are "out of phase" by $\pi/2$. It should be noted that the functions cosine and sine are even and odd functions, respectively, of their arguments.

In some applications it is of interest to plot the absolute values of the cosine and sine functions in polar coordinates. These graphs are shown as Figs. 13a and 13b, respectively.

From Eqs. (36) and (37) it is not difficult to derive the well-known relation

$$sin^2\,\varphi + cos^2\,\varphi = 1, \tag{38}$$

which is applicable for all values of φ. Dividing each term by $cos^2\,\varphi$ leads to the expression

$$\frac{sin^2\,\varphi}{cos^2\,\varphi} + 1 = \frac{1}{cos^2\,\varphi} \tag{39}$$

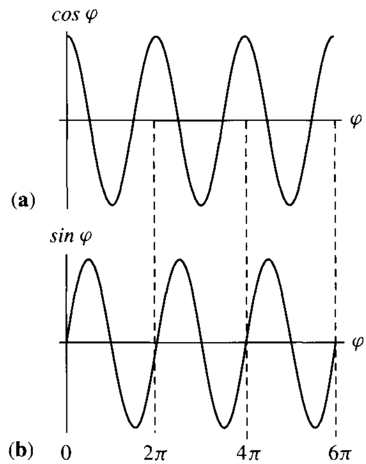

Fig. 12 The functions (**a**) cosine and (**b**) sine.

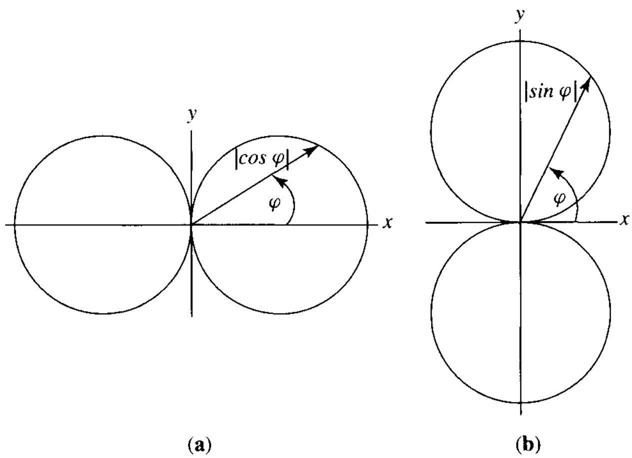

Fig. 13 Absolute values of (**a**) cosine and (**b**) sine in polar coordinates.

or,

$$tan^2 \varphi + 1 = sec^2 \varphi. \tag{40}$$

Similarly, it is easily found that

$$1 + \frac{cos^2 \varphi}{sin^2 \varphi} = \frac{1}{sin^2 \varphi} \tag{41}$$

and thus

$$1 + cot^2\,\varphi = csc^2\,\varphi. \tag{42}$$

Equations (40) and (42) define the trigonometric functions tangent (*tan*), cotangent (*cot*), secant (*sec*) and cosecant (*csc*). These relations were probably learned in high school, but are in any case available on most present-day calculators. The various formulas for more complicated arguments of the trigonometric functions can all be derived from the definitions given in Eqs. (36) and (37). For example, the relation involving the arguments α and β,

$$cos(\alpha - \beta) = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta \tag{43}$$

can be obtained without too much difficulty.

1.8 HYPERBOLIC FUNCTIONS

The trigonometric functions developed in the previous section are referred to as circular functions, as they are related to the circle shown in Fig. 11. Another somewhat less familiar family of functions, the hyperbolic functions, can also be derived from the exponential. They are analogous to the circular functions considered above and can be defined by the relations

$$cosh\,\varphi = \frac{e^{\varphi} + e^{-\varphi}}{2} \tag{44}$$

and

$$sinh\,\varphi = \frac{e^{\varphi} - e^{-\varphi}}{2} \tag{45}$$

The first of these functions is effectively the sum of two simple exponentials, as shown in Fig. 14a, while the hyperbolic sine (*sinh*) is the difference [Eq. (44) and Fig. 14b]. It should be noted that the hyperbolic functions have no real period. They are periodic in the imaginary argument $2\pi i$.

The hyperbolic and circular functions are related *via* the expressions

$$cosh\,\varphi = cos\,i\varphi, \quad cos\,\varphi = cosh\,i\varphi \tag{46}$$

and

$$sinh\,\varphi = \frac{1}{i}\,sin\,i\varphi, \quad sin\,\varphi = \frac{1}{i}\,sinh\,i\varphi. \tag{47}$$

Because of this duality, every relation involving circular functions has its formal counterpart in the corresponding hyperbolic functions, and *vice versa*.

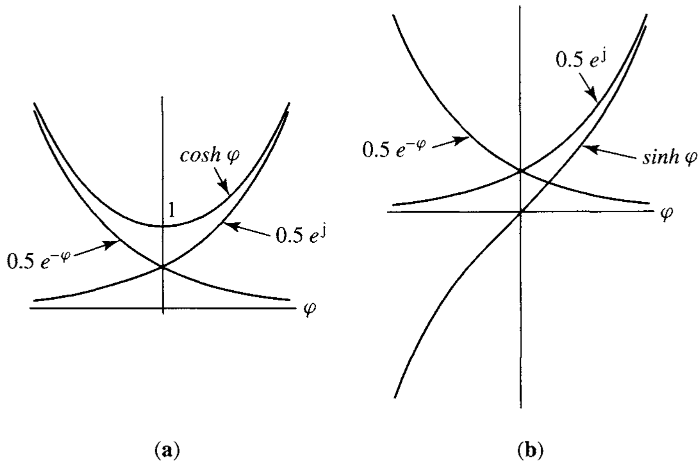

Fig. 14 The hyperbolic functions (**a**) *cosh* and (**b**) *sinh*.

Thus, the various relations between the hyperbolic functions can be derived as carried out above for the circular functions. For example,

$$cosh^2\varphi - sinh^2\varphi = 1, \tag{48}$$

which is analogous to Eq. (38), as illustrated by problem 21.

PROBLEMS

1. Given the decimal 0.7171... , find two numbers whose ratio yields the same value. Ans. 71, 99

2. Repeat question 1 for the decimal 18.35912691269126... . Ans. 18357291, 999900

3. The fraction 22/7 is often used to approximate the value of π. Calculate the error resulting from the use of this approximation. Ans. 0.04%

4. Calculate the values of the expression $\sqrt{log_3 3}$. Ans. ± 1

5. Calculate the values of the expressions $log\ 10^{-3}$ and $ln\ 10^{-3}$. Ans. -3, -6.909

6. Calculate the value of the constant a for which the curve $y = ln((5-x)/(8-x)a)$ passes through the point (1,1). Ans. $a = 7e/4$

7. Derive the general relation between the temperature expressed in degrees Fahrenheit* (F) and degrees Celsius† (C). Ans. $F = \frac{9}{5}C + 32$

8. The length ℓ of an iron bar varies linearly with the temperature over a certain range. At 15°C its length is 1 m. Its length increases by 12 μm/°C. Derive the general relation for ℓ as a function of the temperature t.
Ans. $\ell = 12 \times 10^{-6}t + 0.99982$

9. Calculate the rate constant for a first-order chemical reaction which is 90% completed in 10 min [see Eq. (21)]. Ans. 0.23 min^{-1}

10. A laser beam was used to measure light absorption by a bottle of Bordeaux (1988). In the middle of the bottle (diameter D) 60% of the light was absorbed. At the neck of the bottle (diameter d) it was only 27%. Calculate the ratio of the diameters of the bottle, D/d. What approximations were made in this analysis?
Ans. 2.91

11. With a complex number z defined by Eqs. (30) and (31), find an expression for z^{-1}. Ans. $(1/r)(cos\,\varphi - i\,sin\,\varphi)$

12. Find all of the roots of $\sqrt[4]{16}$. Ans. $2, -2, 2i, -2i$

13. Find all of the roots of the equation $x^3 + 27 = 0$. Ans. $-3, \frac{3}{2}(1 \pm i\sqrt{3})$

14. Given $e^x - e^{-x} = 1, e^x > 1$, find x. Ans. $ln[(1 + \sqrt{5})/2]$

15. Derive the expression for $x(y)$, where $y = ln(e^{2x} - 1)$. Ans. $x = ln\sqrt{e^y + 1}$

16. Write the function $(i + 3)/(i - 1)$ in the form $a + bi$, where $i \equiv \sqrt{-1}$ and a and b are real. Ans. $-1 - 2i$

17. Repeat question 16 for the function $((3i - 7)/(i + 4))$.
Ans. $-(25/17) + (19/17)i$

18. Find the absolute value of the function $(2i - 1)/(i - 2)$. Ans. 1

19. Repeat problem 18 for the function $3i/(i - \sqrt{3})$. Ans. 3/2

20. Given the definitions $cos\,\varphi = (e^{i\varphi} + e^{-i\varphi})/2$ and, $sin\,\varphi = (e^{i\varphi} - e^{-i\varphi})/2i$, show that $cos(\varphi + \gamma) = cos\,\varphi\, cos\,\gamma - sin\,\varphi\, sin\,\gamma$ and therefore, $cos[(\pi/2) - \varphi] = sin\,\varphi$.

21. Given the definitions of the functions *sinh* and *cosh*, prove Eq. (48).

22. Show that $sinh^{-1}x = ln(x + \sqrt{x^2 + 1}), \quad x > 0$.

*Daniel Gabriel Fahrenheit, German physicist (1686–1736).

†Anders Celsius, Swedish astronomer and physicist (1701–1744).