Appendix V: The Laplacian Operator in Spherical Coordinates

Spherical coordinates were introduced in Section 6.4. They were defined in Fig. 6-5 and by Eq. (6-54), namely,

$$x = r \sin\theta \cos\varphi, \tag{1}$$

$$y = r \sin\theta \sin\varphi \tag{2}$$

and

$$z = r \cos\theta. \tag{3}$$

Although transformations to various curvilinear coordinates can be carried out relatively easily with the use of the vector relations introduced in Section 5.15, it is often of interest to make the substitutions directly. Furthermore, it is a very good exercise in the manipulation of partial derivatives.

The relations given in Eq. (1) lead directly to the inverse expressions

$$r^2 = x^2 + y^2 + z^2, \tag{4}$$

$$\sin\theta = \frac{\sqrt{x^2 + y^2}}{r} \tag{5}$$

and

$$\tan\varphi = \frac{y}{x}. \tag{6}$$

The necessary derivatives can be evaluated from the above relations. For example, from Eq. (4)

$$\frac{\partial r}{\partial x} = \tfrac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{r} = \sin\theta\cos\varphi. \tag{7}$$

Similarly, Eq. (5) leads to

$$\frac{\partial\theta}{\partial x} = \frac{x\cos^2\theta}{z^2\tan\theta} = \frac{\cos\theta\cos\varphi}{r} \tag{8}$$

and, Eq. (6) to

$$\frac{\partial \varphi}{\partial x} = -\frac{\sin \varphi}{r \sin \theta}. \tag{9}$$

Note that in the derivation of Eqs. (8) and (9) the relation $(d/d\varsigma)\tan \varsigma = \sec^2 \varsigma$ has been employed, where ς can be identified with either θ or φ. The analogous derivatives can be easily derived by the same method. They are:

$$\frac{\partial r}{\partial y} = \sin \theta \sin \varphi, \tag{10}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos \theta \sin \varphi}{r}, \tag{11}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\cos \varphi}{r \sin \theta}, \tag{12}$$

$$\frac{\partial r}{\partial z} = \cos \theta, \tag{13}$$

$$\frac{\partial \theta}{\partial z} = -\frac{\sin \theta}{r} \tag{14}$$

and

$$\frac{\partial \varphi}{\partial z} = 0. \tag{15}$$

The expressions for the various vector operators in spherical coordinates can be derived with the use of the chain rule. Thus, for example,

$$\begin{aligned}\frac{\partial}{\partial x} &= \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \varphi}{\partial x}\right)\frac{\partial}{\partial \varphi} \\ &= \sin \theta \cos \varphi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \varphi}{r}\frac{\partial}{\partial \theta} - \frac{\sin \varphi}{r \sin \theta}\frac{\partial}{\partial \varphi},\end{aligned} \tag{16}$$

with analogous relations for the two other operators. With the aid of these expressions the nabla, ∇, in spherical coordinates can be derived from Eq. (5-46).

To obtain the Laplacian in spherical coordinates it is necessary to take the appropriate second derivatives. Again, as an example, the derivative of Eq. (16) can be written as

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2} = \sin\theta\cos\varphi \Bigg[& \sin\theta\cos\varphi\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial^2}{\partial r\partial\theta} \\
& -\frac{\cos\theta\cos\varphi}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi} + \frac{\sin\varphi}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\Bigg] \\
& + \frac{\cos\theta\cos\varphi}{r}\Bigg[\sin\theta\cos\varphi\frac{\partial^2}{\partial r\partial\theta} + \cos\theta\cos\varphi\frac{\partial}{\partial r} \\
& + \frac{\cos\theta\cos\varphi}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\sin\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} \\
& + \frac{\cos\theta\sin\varphi}{r\sin^2\theta}\frac{\partial}{\partial\varphi}\Bigg] - \frac{\sin\varphi}{r\sin\theta}\Bigg[\sin\theta\cos\varphi\frac{\partial^2}{\partial r\partial\varphi} \\
& - \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial^2}{\partial\varphi\partial\theta} - \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} \\
& - \frac{\sin\varphi}{r\sin\theta}\frac{\partial^2}{\partial\varphi^2} - \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\Bigg]. \qquad (17)
\end{aligned}
$$

The corresponding operators in y and z are derived in the same way. The sum of these three operators yields the Laplacian as

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \qquad (18)
\end{aligned}
$$

or

$$
\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \qquad (19)
$$

Equation (19) is the classic form of this operator in spherical coordinates as given in Eq. (6-55).