Appendix VI: The Divergence Theorem

The divergence theorem, usually attributed to Gauss, provides a relation between a volume V in space and the area S of the surface that bounds it. The theorem can be simply derived from the following argument.

Consider an element of V along the x direction that is bounded by the xy and xz planes, as shown in Fig. 1. The unit vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the outer normals with respect to the ends of the volume element shown. Thus, for any position vector $\boldsymbol{A}$ its components along a particular outer normal are given by $\boldsymbol{A} \cdot \boldsymbol{n}$. Furthermore, its components A_x along the x axis are functions of x. Thus,

$$\int_{x_1}^{x_2} \frac{\partial A_x}{\partial x} \, \mathrm{d}x = A_x \Big|_{x_1}^{x_2}, \tag{1}$$

where x_1 and x_2 are the values of x at which the element intersects the surface S. If the areas of the ends of the elements are $\mathrm{d}a_1$ and $\mathrm{d}a_2$, as indicated,

$$\mathrm{d}y \, \mathrm{d}z = -\mathrm{d}a_1 \, cos(\boldsymbol{n}_1, x) + \mathrm{d}a_2 \, cos(\boldsymbol{n}_2, x), \tag{2}$$

where $(\boldsymbol{n}_1, x)$ and $(\boldsymbol{n}_2, x)$ are the angles between the corresponding outer normals and the x axis.

If Eq. (1) is multiplied by $\mathrm{d}y\mathrm{d}z$, it becomes

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\partial A_x}{\partial x} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \left[A_{x_1} cos(\boldsymbol{n}_1, x) \, \mathrm{d}a_1 + A_{x_2} cos(\boldsymbol{n}_2, x) \, \mathrm{d}a_2\right] \\ &= A_x \, cos(\boldsymbol{n}, x) \, \mathrm{d}a, \end{aligned} \tag{3}$$

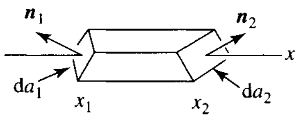

Fig. 1 A volume element in the x direction.

where $\mathrm{d}a = \mathrm{d}a_1 + \mathrm{d}a_2$. The summation of all elements in the x direction leads to

$$\iiint_v \frac{\partial A_x}{\partial x}\,\mathrm{d}v = \iint_S A_x \cos(\boldsymbol{n}, x)\,\mathrm{d}a, \tag{4}$$

with $\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. If this entire procedure is now repeated in the y and z directions, Eq. (4) can be generalized in the form

$$\iiint_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathrm{d}v$$

$$= \iint_S \left[A_x \cos(\boldsymbol{n}, x) + A_y \cos(\boldsymbol{n}, y) + A_z \cos(\boldsymbol{n}, z)\right]\mathrm{d}a, \tag{5}$$

which can be written as

$$\iiint_v \nabla \cdot \boldsymbol{A}\,\mathrm{d}v = \iint_S \boldsymbol{A} \cdot \boldsymbol{n}\,\mathrm{d}a. \tag{6}$$

Equation (6) expresses the divergence theorem.

The divergence theorem has many applications. A very important case is that specified by Eq. (5-66), one of the four equations of Maxwell. It is specifically

$$\nabla \times \mathscr{H} = \boldsymbol{J} + \dot{\mathscr{D}}. \tag{7}$$

It leads directly to the equation of continuity for the charge density in a closed volume, *viz.*

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}, \tag{8}$$

which is Eq. (5-68).

The equation of continuity also finds application in thermodynamics, as the flux density of heat from an enclosed volume must be compensated by a corresponding rate of temperature decrease within. Similarly, in fluid dynamics, if the volume contains an incompressible liquid, the flux density of flow from the volume results in an equivalent rate of decrease in the density within the enclosure.